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THEORIES OF IDEAL PLASTICITY WITH A SINGULAR YIELD SURFACE

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Specific features of the theories of ideal plasticity which are based on the Tresca yield criterion and the maximum reduced stress criterion are discussed. An analysis is carried out in terms of the canonical basis of the deviatoric stress tensor.

1. The following statement is well known. In order that a real symmetric matrix A be orthogonally similar to a matrix whose elements on the main diagonal are all equal to zero, it is necessary and sufficient that the trace of the matrix A vanish. In particular, for a 3×3 matrix, this orthogonal transformation of similarity is not unique and depends on one parameter [1, 2]. We study this question in detail using the results of [1, 2].

Let D denote a symmetric tensor of rank 2 in three-dimensional space with zero trace, and d_{ij} the components of this tensor in an arbitrary Cartesian coordinate system with basis $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, and $\boldsymbol{k}_3$. We consider elements with the unit normal vector $\boldsymbol{n} = (n_1, n_2, n_3)$, for which the condition

$$d_n \equiv d_{ij} n_i n_j = 0$$

holds. Hereafter, summation from 1 to 3 over repeat indices is performed. The set of these elements forms a second-order cone N. For each vector $\boldsymbol{n} \in N$, one can find vectors $\boldsymbol{n}'$, $\boldsymbol{n}'' \in N$ such that the triad $\boldsymbol{n}$, $\boldsymbol{n}'$, and $\boldsymbol{n}''$ forms an orthogonal basis. In this basis, all the diagonal elements of the matrix D vanish [1]. This representation of the deviator is called a canonical representation, and the orthonormal basis $\boldsymbol{n}$, $\boldsymbol{n}'$, and $\boldsymbol{n}''$ is called a canonical basis [2]. We construct examples of these bases.

We denote the basis of the principal axes of the deviator by $\boldsymbol{k}_1^*$, $\boldsymbol{k}_2^*$, and $\boldsymbol{k}_3^*$. In this basis

$$D \sim \begin{Vmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{Vmatrix},$$

the principal values having the form [3]

$$d_1 = d\cos\theta, \quad d_2 = d\cos\left(\theta - \frac{2\pi}{3}\right), \quad d_3 = d\cos\left(\theta - \frac{4\pi}{3}\right),$$

where $d = (2/\sqrt{3})\sqrt{J_2^D}$, $\cos 3\theta = (3\sqrt{3}/2)\, J_3^D/(J_2^D)^{3/2}$, and $J_2^D = (1/2)d_{ij}d_{ij}$ and $J_3^D = (1/3)d_{ij}d_{jk}d_{ki}$ are the second and third invariants of the deviator. It is obvious that

$$\frac{\partial J_2^D}{\partial \sigma_{ij}} = d_{ij}, \qquad \frac{\partial J_3^D}{\partial \sigma_{ij}} = d_{ik}d_{kj} - \frac{2}{3}J_2^D\delta_{ij}, \tag{1.1}$$

where δ_{ij} is the Kronecker delta and $d_{ij} = \sigma_{ij} - (1/3)(\sigma_{kl}\delta_{kl})\delta_{ij}$.

We assume that the orthonormal basis $\boldsymbol{k}_1^{\mathrm{I}}$, $\boldsymbol{k}_2^{\mathrm{I}}$, $\boldsymbol{k}_3^{\mathrm{I}}$ is related to the basis $\boldsymbol{k}_1^*$, $\boldsymbol{k}_2^*$, $\boldsymbol{k}_3^*$ by means of the matrix $M = \|m_{ij}\|$:

$$\boldsymbol{k}_i^{\mathrm{I}} = m_{ij}\boldsymbol{k}_j^*,$$

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where $m_{11} = m_{12} = m_{13} = 1/\sqrt{3}$, $m_{21} = \sqrt{2/3}\cos\alpha$, $m_{22} = \sqrt{2/3}\cos(\alpha - 2\pi/3)$, $m_{23} = \sqrt{2/3}\cos(\alpha - 4\pi/3)$, $m_{31} = \sqrt{2/3}\cos\beta$, $m_{32} = \sqrt{2/3}\cos(\beta - 2\pi/3)$, $m_{33} = \sqrt{2/3}\cos(\beta - 4\pi/3)$, $\alpha = -\theta/2 - \pi/4$, and $\beta = -\theta/2 + \pi/4$. In this basis, we have

$$D \sim \frac{\sqrt{J_2^D}}{\sqrt{3}} \begin{Vmatrix} 0 & \sqrt{2}\cos\left(\frac{3}{2}\theta + \frac{\pi}{4}\right) & \sqrt{2}\sin\left(\frac{3}{2}\theta + \frac{\pi}{4}\right) \\ \sqrt{2}\cos\left(\frac{3}{2}\theta + \frac{\pi}{4}\right) & 0 & 1 \\ \sqrt{2}\sin\left(\frac{3}{2}\theta + \frac{\pi}{4}\right) & 1 & 0 \end{Vmatrix}.$$

We shall call this representation of the deviator a *basic canonical* representation, and the basis $\boldsymbol{k}_1^{\mathrm{I}}$, $\boldsymbol{k}_2^{\mathrm{I}}$, and $\boldsymbol{k}_3^{\mathrm{I}}$ a *basic canonical basis.*

Obviously, rotating the basis about the principal axes through the angle $\pm\pi/2$, one can reduce the deviator to the form

$$D \sim \begin{Vmatrix} d_{\max} & 0 & 0 \\ 0 & d_{\mathrm{mean}} & 0 \\ 0 & 0 & d_{\min} \end{Vmatrix},$$

where $|d_{\max}| \geqslant |d_{\mathrm{mean}}| \geqslant |d_{\min}|$ and $d_{\max} + d_{\mathrm{mean}} + d_{\min} = 0$. We shall call the above system of the principal axes of the deviator D an ordered system and denote its orthonormal basis by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$.

We consider the basis $\boldsymbol{e}_1^{\mathrm{I}}$, $\boldsymbol{e}_2^{\mathrm{I}}$, $\boldsymbol{e}_3^{\mathrm{I}}$ defined as follows:

$$\boldsymbol{e}_i^{\mathrm{I}} = l_{ij}^{\mathrm{I}}\boldsymbol{e}_j, \qquad L^{\mathrm{I}} = \|l_{ij}^{\mathrm{I}}\|,$$

$$L^{\mathrm{I}} = \begin{Vmatrix} \alpha & \beta & 0 \\ \beta/\sqrt{2} & -\alpha/\sqrt{2} & -1/\sqrt{2} \\ -\beta/\sqrt{2} & \alpha/\sqrt{2} & -1/\sqrt{2} \end{Vmatrix}, \qquad \begin{aligned} \alpha &= \sqrt{-d_{\mathrm{mean}}/(d_{\max} - d_{\mathrm{mean}})}, \\ \beta &= \sqrt{d_{\max}/(d_{\max} - d_{\mathrm{mean}})}. \end{aligned}$$

Using the transformation formulas $d_{ij}^{\mathrm{I}} = l_{ip}^{\mathrm{I}} l_{jq}^{\mathrm{I}} \hat{d}_{pq}$, where $\hat{d}_{11} = d_{\max}$, $\hat{d}_{22} = d_{\mathrm{mean}}$, $\hat{d}_{33} = d_{\min}$, and $\hat{d}_{12} = \hat{d}_{13} = \hat{d}_{23} = 0$, we infer that, in the basis $\boldsymbol{e}_1^{\mathrm{I}}$, $\boldsymbol{e}_2^{\mathrm{I}}$, $\boldsymbol{e}_3^{\mathrm{I}}$, the deviator D reduces to the form

$$D \sim \begin{Vmatrix} 0 & b & -b \\ b & 0 & a \\ -b & a & 0 \end{Vmatrix}, \qquad \begin{aligned} &b = \varepsilon\sqrt{-d_{\mathrm{mean}} d_{\max}/2}, \\ &a = d_{\min}, \quad \varepsilon = \operatorname{sign}(d_{\max}). \end{aligned}$$

We shall call this representation of the deviator the first canonical representation, and the orthonormal basis $\boldsymbol{e}_1^{\mathrm{I}}$, $\boldsymbol{e}_2^{\mathrm{I}}$, $\boldsymbol{e}_3^{\mathrm{I}}$ the first canonical basis.

We consider the orthogonal basis $\boldsymbol{e}_1^{\varphi}$, $\boldsymbol{e}_2^{\varphi}$, $\boldsymbol{e}_3^{\varphi}$:

$$\begin{gathered} \boldsymbol{e}_i^{\varphi} = l_{ij}^{\varphi}\boldsymbol{e}_j^{\mathrm{I}}, \quad L^{\varphi} = \|l_{ij}^{\varphi}\|, \quad l_{11}^{\varphi} = \cos\varphi\cos\psi + \sin\varphi\sin\psi\sin\chi, \quad l_{12}^{\varphi} = \sin\psi\cos\chi, \\ l_{13}^{\varphi} = -\sin\varphi\cos\psi + \cos\varphi\sin\psi\sin\chi, \quad l_{21}^{\varphi} = -\cos\varphi\sin\psi + \sin\varphi\cos\psi\sin\chi, \\ l_{22}^{\varphi} = \cos\psi\cos\chi, \qquad l_{23}^{\varphi} = \sin\varphi\sin\psi + \cos\varphi\cos\psi\sin\chi, \\ l_{31}^{\varphi} = \sin\varphi\cos\chi, \qquad l_{32}^{\varphi} = -\sin\chi, \qquad l_{33}^{\varphi} = \cos\varphi\cos\chi. \end{gathered} \tag{1.2}$$

Here, we have $0 \leqslant \varphi < 2\pi$,

$$\begin{gathered} \tan\chi = -\frac{\sin\varphi}{\omega + \tan\varphi}, \quad \cot 2\psi = \frac{\cos^3\varphi\sin^2\chi - \cos\varphi\sin^2\varphi - \cos\chi\sin\chi}{2\cos^2\varphi\sin\varphi\sin\chi}, \\ \omega = \frac{\varepsilon\sqrt{2}d_{\min}}{\sqrt{-d_{\max}d_{\mathrm{mean}}}}, \qquad \varepsilon = \operatorname{sign}(d_{\max}). \end{gathered} \tag{1.3}$$

In the orthonormal basis e_1^φ, e_2^φ, and e_3^φ, for any $0 \leqslant \varphi < 2\pi$ the deviator D has the form

$$D \sim \begin{Vmatrix} 0 & d_{12}^\varphi & d_{13}^\varphi \\ d_{12}^\varphi & 0 & d_{23}^\varphi \\ d_{13}^\varphi & d_{23}^\varphi & 0 \end{Vmatrix}; \tag{1.4}$$

$$(d_{12}^\varphi)^2 + (d_{13}^\varphi)^2 + (d_{23}^\varphi)^2 = (J_2^D)^2, \qquad d_{12}^\varphi d_{13}^\varphi d_{23}^\varphi = J_3^D/2. \tag{1.5}$$

We consider the basis $e_1^{\rm II}$, $e_2^{\rm II}$, and $e_3^{\rm II}$ defined as follows:

$$e_i^{\rm II} = l_{ij}^{\rm II} e_j, \qquad L^{\rm II} = \|l_{ij}^{\rm II}\|,$$

$$L^{\rm II} = \begin{Vmatrix} \gamma & 0 & \delta \\ \delta/\sqrt{2} & 1/\sqrt{2} & -\gamma/\sqrt{2} \\ -\delta/\sqrt{2} & 1/\sqrt{2} & \gamma/\sqrt{2} \end{Vmatrix}, \qquad \begin{matrix} \gamma = \sqrt{-d_{\min}/(d_{\max} - d_{\min})}, \\ \delta = \sqrt{d_{\max}/(d_{\max} - d_{\min})}. \end{matrix}$$

In the basis $e_1^{\rm II}$, $e_2^{\rm II}$, $e_3^{\rm II}$, the deviator D reduces to the form

$$D \sim \begin{Vmatrix} 0 & c & -c \\ c & 0 & f \\ -c & f & 0 \end{Vmatrix}, \qquad \begin{matrix} c = \varepsilon\sqrt{-d_{\max} d_{\min}/2}, \\ \varepsilon = \operatorname{sign}(d_{\max}) = \pm 1, \\ f = d_{\rm mean}. \end{matrix}$$

We shall call this representation of the deviator the second canonical representation, and the orthonormal basis $e_1^{\rm II}$, $e_2^{\rm II}$, $e_3^{\rm II}$ the second canonical basis.

Obviously, the orthogonal family of canonical bases constructed above can also be obtained by replacing ω in (1.3) by $\omega_1 = \varepsilon\sqrt{2} d_{\rm mean}/\sqrt{-d_{\max} d_{\min}}$, where $\varepsilon = \operatorname{sign}(d_{\max})$, and $e_j^{\rm I}$ in (1.2) by $e_j^{\rm II}$, where $j = 1, 2, 3$.

2. Let S be the deviatoric stress tensor, and e_1, e_2, and e_3 its arbitrary canonical basis, in which the deviator takes the form

$$S \sim \begin{Vmatrix} 0 & s_{12} & s_{13} \\ s_{12} & 0 & s_{23} \\ s_{13} & s_{23} & 0 \end{Vmatrix}.$$

The Mises yield criterion $J_2^s = k_*^2$, where k_* is the yield constant, is the sphere in the space of s_{12}, s_{13}, and s_{23}:

$$s_{12}^2 + s_{13}^2 + s_{23}^2 = k_*^2.$$

The Tresca yield criterion [4] has the form

$$8(2k_*^2 - J_2^s)^3 - 4(J_2^s)^2(3k_*^2 - J_2^s) + 27(J_3^s)^2 = 0; \tag{2.1}$$

$$J_2^s = s_{12}^2 + s_{13}^2 + s_{23}^2, \qquad J_3^s = 2 s_{12} s_{13} s_{23}. \tag{2.2}$$

The convex domain bounded by this surface is shown in Fig. 1 in the x, y, and z axes, where

$$x = s_{12}/k_*, \qquad y = s_{13}/k_*, \qquad z = s_{23}/k_*. \tag{2.3}$$

The Mises yield criterion in these axes represents a unit sphere inscribed into the Tresca surface. The points at which $x^2 = y^2 = z^2$ are the singular (conical) points. In particular, at the point $x = y = z = 2/3$, the axis of the circular cone of the tangents of the opening span $2\arctan(3/\sqrt{2}) \simeq 130°$ is equally inclined to the coordinate axes. We note that the stress state which corresponds to the singular points has equal, in absolute value, shear stresses:

$$|s_{12}| = |s_{13}| = |s_{23}| = 2k_*/3. \tag{2.4}$$

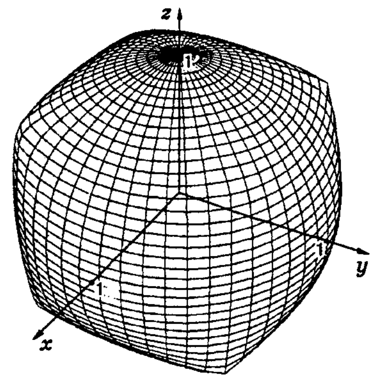

Fig. 1

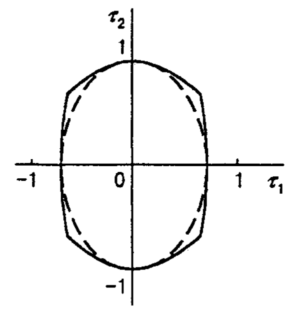

Fig. 2

The following parametrization is valid for the Tresca surface, excluding its singular points:

$$s_{12} = k_* r(\psi, \theta) \cos\psi \cos\theta, \quad -\pi/2 \leqslant \theta \leqslant \pi/2, \quad 0 \leqslant \psi < 2\pi, \quad s_{13} = k_* r(\psi, \theta) \sin\psi \cos\theta,$$
$$\theta \neq \pm\arccos(1/\sqrt{3}), \quad s_{23} = k_* r(\psi, \theta) \sin\theta, \quad \psi \neq (\pi/4)j, \quad j = 1,\, 3,\, 5,\, 7, \tag{2.5}$$

where

$$r(\psi, \theta) = \Big(\frac{3}{a} - \frac{-81 - 72a}{9a(-27 - 36a - 8a^2 + 8a^{3/2}\sqrt{1+a})^{1/3}} + \frac{(-27 - 36a - 8a^2 + 8a^{3/2}\sqrt{1+a})^{1/3}}{a}\Big)^{1/2},$$
$$a = 27(\cos^2\psi \sin^2\psi \cos^4\theta \sin^2\theta) - 1; \quad -1 \leqslant a < 0.$$

In the first and second canonical bases, the deviatoric stress tensor has the form

$$S \sim \begin{Vmatrix} 0 & \tau_1 & -\tau_1 \\ \tau_1 & 0 & \tau_2 \\ -\tau_1 & \tau_2 & 0 \end{Vmatrix}.$$

The Tresca yield criterion (2.1) in the $(\tau_1,\, \tau_2)$ plane is written as follows:

$$(-2k_*^2 + \tau_1^2 - 3\tau_2 k_* - \tau_2^2)(-2k_*^2 + \tau_1^2 + 3\tau_2 k_* - \tau_2^2)(8\tau_1^2 + \tau_2^2 - 4k_*^2) = 0.$$

This piecewise-smooth curve, which consists of the parts of an ellipse (the lateral sides) and the parts of the hyperbolas, is shown in Fig. 2 for $k_* = 1$ by the solid curve, and the dashed curve refers to the Mises ellipse:

$$2\tau_1^2 + \tau_2^2 = k_*^2.$$

We denote the plastic strain-rate tensor by $\dot{E}^p$ and its components in the canonical basis of the tensor S by ξ_{ij}; let

$$\xi_{11} + \xi_{22} + \xi_{33} = 0. \tag{2.6}$$

Assuming that the associated law of plastic flow is valid, using the right-hand side of (2.1) as the plastic potential, and bearing in mind (1.1), we obtain

$$\xi_{11} = \frac{\Lambda}{3}\nu(s_{12}^2 + s_{13}^2 - 2s_{23}^2), \qquad \begin{smallmatrix} & 1 & \\ \nearrow & & \searrow \\ 3 & \longleftarrow & 2 \end{smallmatrix} \qquad \xi_{12} = \Lambda(\mu s_{12} + \nu s_{13}s_{23}), \tag{2.7}$$

where Λ is an arbitrary factor, the symbol $\begin{smallmatrix} & 1 & \\ \nearrow & & \searrow \\ 3 & \longleftarrow & 2 \end{smallmatrix}$ denotes the cyclic permutation of the indices, $\mu = -12[(J_2^s)^2 - 6k_*^2 J_2^s + 8k_*^4]$, and $\nu = 54J_3^s$.

At the nonsingular points of the Tresca surface (2.1), the condition (2.4) is not satisfied and, therefore, the components of the plastic strain tensor ξ_{11}, ξ_{22}, ξ_{33} cannot vanish simultaneously. This means that the canonical basis of the stress tensor does not coincide with that of the plastic strain tensor. In this case, according to equality (1.1), the tensors S and $\dot{E}^p$ are coaxial. It follows from (2.5) and (2.7) that, for all nonsingular points at the Tresca surface, the following condition holds:

$$\det \|\xi_{ij}\| = 0. \tag{2.8}$$

Hence, one of the eigenvalues of the tensor $\dot{E}^p$ is zero and the other two eigenvalues are ω and $-\omega$ by virtue of (2.6), where $\omega \neq 0$ by virtue of (2.7); therefore, the plane-strain state is realized [5]. Thus, the tensor $(\dot{E}^p)^2$ has a nonmultiple zero eigenvalue and a multiple eigenvalue ω^2. Let $\boldsymbol{r}$ be the unit eigenvector of the tensor $(\dot{E}^p)^2$ which corresponds to the zero eigenvalue. The representation [6]

$$(\dot{E}^p)^2 = \omega^2(I - \boldsymbol{r} \otimes \boldsymbol{r}) \tag{2.9}$$

is valid. Here I is the identity tensor and the symbol $\otimes$ denotes the dyadic product.

At the singular points of the Tresca surface, the condition (2.4) is satisfied; therefore, two of the principal stresses coincide and, consequently, only one of the principal axes S is defined uniquely. We require that this principal axis coincide with one of the principal axes of the tensor $\dot{E}^p$, which is the condition that relates the tensors S and $\dot{E}^p$ at the singular points of the Tresca surface. As an example, we consider the case

$$s_{12} = s_{13} = s_{23} = 2k_*/3. \tag{2.10}$$

The eigenvalues of the tensor S are $\lambda_1^s = 4k_*/3$, $\lambda_2^s = -2k_*/3$, and $\lambda_3^s = -2k_*/3$. The eigenvector which corresponds to the eigenvalue λ_1^s has the form $\boldsymbol{n} = (1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3})$. For the tensor S having a multiple eigenvalue, the following representation [6] is valid:

$$S = 2k_*\Big(\boldsymbol{n} \otimes \boldsymbol{n} - \frac{1}{3}I\Big). \tag{2.11}$$

Let λ be the nonmultiple eigenvalue of the tensor $\dot{E}^p$; then from the equality

$$\dot{E}^p \cdot \boldsymbol{n} = \lambda \boldsymbol{n} \tag{2.12}$$

we have $\xi_{11} + \xi_{12} + \xi_{13} = \lambda$, $\xi_{12} + \xi_{22} + \xi_{23} = \lambda$, and $\xi_{13} + \xi_{23} + \xi_{33} = \lambda$. The condition of the positive dissipation $s_{ij}\xi_{ij} > 0$ and (2.8) imply that $\lambda > 0$.

The equalities (2.11) and (2.12) are valid in an arbitrary coordinate system and they were derived using the different assumptions in [5].

We note that, in the case defined by the equality (2.10), the relations $|\lambda_1^s - \lambda_2^s| = 2k_*$ and $|\lambda_1^s - \lambda_3^s| = 2k_*$ are valid. This means that the Haar–von Kármán condition of the complete yielding [7] is realized, which holds also at all the singular points [see (2.4)] of the Tresca surface.

3. We examine the question as to when the conditions of complete plasticity (2.10) are realized in a certain domain of a body. In view of the equality (2.11), the stress tensor in an arbitrary Cartesian coordinate system $(x_1,\ x_2,\ x_3)$ becomes $\sigma_{ij} = 2k_*(p\delta_{ij} + n_i n_j)$. Using the equilibrium equations, we have the following system to determine the functions $p = p(x_1, x_2, x_3)$ and $n_i = n_i(x_1, x_2, x_3)$, where $i = 1,\ 2,\ 3$:

$$p_{,i} + n_j n_{i,j} + n_i n_{s,s} = 0 \qquad (i,\ j,\ s = 1,\ 2,\ 3); \tag{3.1}$$

$$n_1^2 + n_2^2 + n_3^2 = 1. \tag{3.2}$$

Here and henceforth, the comma denotes the derivative with respect to the coordinates. A series of properties of system (3.1), (3.2) was considered in [5], and the group analysis of this system by the Lie–Ovsyannikov method [8] was performed in [9]. We give the complete description of the partially invariant solution, for which

$$n_i = N_i(p), \qquad p = p(x_1, x_2, x_3), \qquad i = 1,\ 2,\ 3. \tag{3.3}$$

Substituting (3.3) into (3.1), we obtain

$$(p\delta_{ij} + N_i N_j)' p_{,i} = 0. \tag{3.4}$$

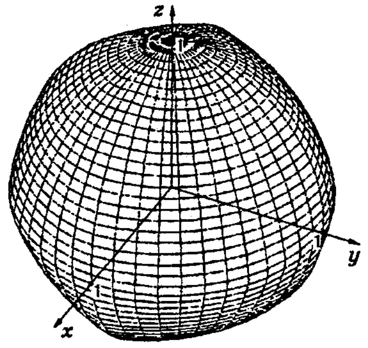

Fig. 3

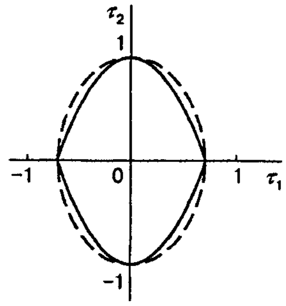

Fig. 4

Hereafter, the prime denotes the derivative with respect to p. The compatibility condition of system (3.4) implies that

$$(N_1')^2 + (N_2')^2 + (N_3')^2 = 1. \tag{3.5}$$

Let $M = M(t)$ ($0 \leqslant t \leqslant T$ be an arbitrary, twice differentiable function; moreover, $M^2 + \dot{M}^2 < 1$ and $M + \ddot{M} \neq 0$. Here and henceforth, the dot denotes differentiation with respect to t. Relations (3.2) and (3.5) are satisfied if one assumes that

$$N_1 = \nu_1(t) = \dot{M}\cos t + M\sin t, \qquad N_2 = \nu_2(t) = \dot{M}\sin t - M\cos t,$$
$$N_3 = \nu_3(t) = (1 - M^2 - \dot{M}^2)^{1/2}, \qquad p = P(t) = \int_0^t \frac{(M + \ddot{M})(1 - M^2)^{1/2}}{(1 - M^2 - \dot{M}^2)^{1/2}}\,dt. \tag{3.6}$$

Introducing (3.6) into (3.1) and integrating the first two equations, we obtain

$$x_1 l_1(t) + x_2 l_2(t) + x_3 l_3(t) = g(t), \tag{3.7}$$

where

$$l_1(t) = (\nu_1\nu_3)^{\cdot}(P + \nu_2^2)^{\cdot} - (\nu_1\nu_2)^{\cdot}(\nu_2\nu_3)^{\cdot}, \quad l_2(t) = (\nu_2\nu_3)^{\cdot}(P + \nu_1^2)^{\cdot} - (\nu_1\nu_2)^{\cdot}(\nu_1\nu_3)^{\cdot},$$
$$l_3(t) = -(P + \nu_1^2)^{\cdot}(P + \nu_2^2)^{\cdot} + (\nu_1\dot{\nu}_2 + \dot{\nu}_2\nu_1)^2,$$

and $g(t)$ is an arbitrary differentiable function. The equality (3.7) defines implicitly the function $t = t(x_1, x_2, x_3)$.

The constructed solution involves two arbitrary functions of one variable.

If one sets $t = t_0 = \text{const}$ in (3.7), this equation defines a plane in the (x_1, x_2, x_3) space on which all the components of the stress tensor are constant. In a particular case where $g(p) \equiv 0$, this plane passes through the coordinate origin. This solution can be used to describe the limiting state of a body bounded by the planes.

4. We consider the yield criterion of the maximum reduced stress [10] $|s_{\max}| = k_*$, where $s_{\max}$ is the maximum absolute value of the eigenvalues of the deviatoric stress tensor S. Substituting k_*^2 into the cubic equation, whose roots are the squared eigenvalues of S, we obtain

$$(J_3^S + J_2^S k_* - k_*^3)(J_3^S - J_2^S k_* + k_*^3) = 0. \tag{4.1}$$

The convex domain bounded by this surface is shown in Fig. 3 in the x, y, and z axes [see (2.3)]. The surface (4.1) is inscribed into the Mises sphere. The points at which this surface cuts the coordinate axes are singular. The conical surfaces which correspond to these points split up into pairs of planes.

In the first and second canonical bases [see (2.5)], the criterion of the maximum reduced stress assumes the form

$$(2\tau_1^2\tau_2 + 2\tau_1^2 k_* + \tau_2^2 k_* - k_*^3)(2\tau_1^2\tau_2 - 2\tau_1^2 k_* - \tau_2^2 k_* + k_*^3) = 0.$$

The given piecewise-smooth curve (for $k_* = 1$) is shown in Fig. 4. The dashed curve refers to the Mises ellipse.

The associated law of plastic flow with potential (4.1) leads to the equalities (2.7), where $\mu = -2k_*(J_2^S - k_*^2)$ and $\nu = 2J_3^S$.

For the nonsingular points of the surface (4.1), the tensors S and $\dot{E}^p$ are coaxial, and, for the nonsingular points at which the condition $|s_{12}| = |s_{13}| = |s_{23}| = k_*$ is satisfied, the canonical basis of the stress tensor is also the canonical basis of the plastic strain tensor.

We now establish the relation between the tensors S and $\dot{E}^p$ at the singular points of the surface (4.1). As an example, we consider the case where $s_{12} = k_*$; $s_{13} = 0$; $s_{23} = 0$. The eigenvalues of the tensor S are k_*, $-k_*$, and 0.

We assume that the plastic strain rate tensor $\dot{E}^p$ satisfies the incompressibility condition (2.6) and is coaxial with the tensor S. The representation [11] is then valid

$$\dot{E}^p = \alpha S + \beta\Big(S^2 - \frac{2k_*^2}{3} I\Big),$$

where α and β are arbitrary constants, and $\alpha > 0$ by virtue of the condition of the positive dissipation.

In concluding, we note that the detailed description of the set of deviators with one common principal direction (see Sec. 2) can be found in [12].

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